This unit will show you how to:

- Draw and interpret graphs modelling real-life situations.
- Determine the maximum and minimum values of a function.
- Determine increase and decrease of a function.
- Recognise continuous and discontinuous functions.
- Identify periodic functions.
- Match analytical expressions and graphs and vice versa.

<table>
<thead>
<tr>
<th>Keywords</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Graph</td>
<td>Increase, decrease</td>
</tr>
<tr>
<td>Function</td>
<td>Maximum, minimum</td>
</tr>
<tr>
<td>Independent variable</td>
<td>Periodic function</td>
</tr>
<tr>
<td>Dependent variable</td>
<td>Continuous function</td>
</tr>
<tr>
<td>Domain, range</td>
<td>Analytical expression</td>
</tr>
</tbody>
</table>

7.1.- FUNCTIONS AND THEIR GRAPHS

In everyday life, many quantities depend on one or more changing variables. For example:

a) Plant growth depends on sunlight and rainfall.

b) Distance travelled depends on speed and time taken.

c) Test marks depend on attitude, listening in lectures and doing tutorials (among many other variables!!).

Graphs can be used to quickly get an idea of how one quantity varies as another quantity changes. Let’s see two examples:

1) This graph shows Dan’s journey on his bike:
2) You can use a graph to **model** the depth of water flowing in or out of a container at a constant rate.

![Image of water container and graph](image)

**Exercise 1**

The distance-time graph shows the journey of a car between Birmingham and Stoke-on-Trent.

![Distance-time graph](image)

a) How far is it from Birmingham to Stoke-on-Trent?
b) For how long did the car stop?
c) What has the speed of the car for the first part of the journey?
d) Between which two times was the car travelling fastest?
e) What was the average speed of the car for the whole journey?

**Exercise 2**

Sketch a graph to show what happen as

1. Sam fills a bath with both taps running.
2. He realises it is too hot so turns off the hot tap.
3. He turns off the cold tap.
4. He gets in.
5. Has a long soak.
7. Pulls out the plug.
Definition of a function

A function is a relation between two variables such that for every value of the first, there is only one corresponding value of the second. We say that the second variable is a function of the first variable.

The first variable is the independent variable (usually x), and the second variable is the dependent variable (usually y).

The independent variable and the dependent variable are real numbers.

Examples:

1) You know the formula for the area of a circle A = πr². This is a function as each value of the independent variable r gives you one value of the dependent variable A.

2) The force F required to accelerate an object of mass 5 kg by an acceleration of a is given by: F = 5a. Here, F is a function of the acceleration, a. The dependent variable is F and the independent variable is a.

3) In the equation y = x², y is a function of x, since for each value of x, there is only one value of y.

We normally write functions as f(x), and read this as "function f of x". For example, the function y = x² − 5x + 6 is also written as f(x) = x² − 5x + 6. (y and f(x) are the same).

The value of the function f(x) when x = a is f(a).

If f(x) = x² − 5x + 6, then f(3) = 3² − 5 · 3 + 6 = 9 − 15 + 6 = 0.

Graphical representation

A good way of presenting a function is by graphical representation. Graphs give us a visual picture of the function.

Normally, the values of the independent variable (generally the x-values) are placed on the horizontal axis, while the values of the dependent variable (generally the y-values) are placed on the vertical axis.
The x-value, called the **abscissa**, is the perpendicular distance of P from the y-axis.

The y-value, called the **ordinate**, is the perpendicular distance of P from the x-axis.

The values of x and y together, written as (x,y) are called the **coordinates** of the point P.

**Domain and range**

The **domain** of a function is the complete set of possible values of the independent variable in the function.

The **range** of a function is the complete set of all possible resulting values of the dependent variable of a function, after we have substituted the values in the domain.

**Example**: Find the domain and the range for the function \( y = \sqrt{x} + 4 \).

**Exercise 3**

Find the domain and the range for the following functions:

**A**

**B**

---

Mathematics 3° ESO. IES Don Bosco (Albacete). European Section
7.2.- VARIATIONS IN A FUNCTION

Increasing and decreasing

A function \( f \) is \textbf{increasing} on an interval \((a, b)\) if for any \( x_1 \) and \( x_2 \) in the interval such that \( x_1 < x_2 \) then \( f(x_1) < f(x_2) \). Another way to look at this is: as you trace the graph from \( a \) to \( b \) (that is from left to right) the graph should go up.

A function \( f \) is \textbf{decreasing} on an interval \((a, b)\) if for any \( x_1 \) and \( x_2 \) in the interval such that \( x_1 < x_2 \) then \( f(x_1) > f(x_2) \). Another way to look at this is: as you trace the graph from \( a \) to \( b \) (that is from left to right) the graph should go down.

Maxima and minima

A function \( f \) has a \textbf{relative (or local) maximum} at a point if its ordinate is greater than the ordinates of the points around it.

A function \( f \) has a \textbf{relative (or local) minimum} at a point if its ordinate is less than the ordinates of the points around it.
A function $f$ has an **absolute (or global) maximum** at a point if its ordinate is the largest value that the function takes on the domain that we are working on.

A function $f$ has an **absolute (or global) minimum** at a point if its ordinate is the smallest value that the function takes on the domain that we are working on.

**Example:**

The following graph shows the present people (in thousands) in a shopping centre during a day.

The function increases in the intervals $(9, 12)$ and $(14, 18)$.  
The function decreases in the intervals $(12, 14)$ and $(18, 24)$.

The point $(12, 3)$ is a local maximum and the point $(14, 2)$ is a local minimum.

At 09:00 the function has the global minimum and at 18:00 the function has the global maximum (5000 people).
**Exercise 4**

The following graph shows the number of people who have lived in a city during some years.

Determine the intervals of increasing and decreasing of this function as well as its maxima and minima.

**Exercise 5**

Draw the graph of a function with the following features:

a) Local maxima at $x = -2$ and $x = 3$.

b) Local minima at $x = 1$ and $x = 2$.

c) A global maximum at $x = 0$.

d) A global minimum at $x = -1$.

**7.3.- TENDENCIES IN A FUNCTION**

Do you remember what the relationship between pressure and volume is?

For a fixed amount of any gas, kept a fixed temperature, pressure and volume are linked by the formula $p = \frac{k}{v}$, where $p$ is the pressure, $v$ is the volume and $k$ is a constant. A gas cannot have a negative volume, so the values of $v$ will be greater than 0.

The graph for this formula is shown opposite:

The pressure of the gas gets higher as the volume gets smaller.

As the volume increases, the pressure gets close to zero. As the volume gets closer to zero, the pressure gets higher, but the volume never actually equals zero.
There are functions where, although you only know a piece of them, you can guess how they will work far away from that piece. As you have seen in the previous example, these functions have “branches” with a very clear tendency.

**Periodic functions**

The following graph shows the distance from the Halley’s Comet to the Sun during the last two centuries.

![Graph showing distance to the Sun over time](image)

Since the orbit is repeated over and over every 76 years, the graph is also repeated in this period of time. This is a periodic function with period 76.

A **periodic function** is a function which has a graph that repeats itself identically over and over as it is followed from left to right.

The horizontal distance required for the graph of a periodic function to complete one cycle is called **period**.

**Exercise 6**

*When a big wheel turns, its baskets go up and go down. This is the graphical representation of the function time-distance to the floor of one of these baskets.*

![Graph showing time-distance to the floor of a basket](image)

a) How long does the basket take to complete a turn?  
b) What is the maximum height? What is the radius of the big wheel?  
c) Explain how to calculate the height at 130 seconds without going on the graph.
**Exercise 7**

Which of the following functions are periodic? What are their periods?

![Graphs](image)

### 7.4. - CONTINUOUS AND DISCONTINUOUS FUNCTIONS

**Example 1:** Steam in a boiler was heated to 150°C. Its temperature was then recorded each minute as follows:

<table>
<thead>
<tr>
<th>Time (min)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temp (°C)</td>
<td>150.0</td>
<td>142.8</td>
<td>138.5</td>
<td>135.2</td>
<td>132.7</td>
<td>130.8</td>
</tr>
</tbody>
</table>

Since the temperature changes in a continuous way, there is meaning to the values in the intervals between the points. Therefore, the points are joined by a smooth curve. This is a continuous function.

![Graph](image)

The graph of a **continuous function** can be drawn without lifting the pencil from the paper.

**Example 2:** The table below shows the postal rates in the U.K. for first class letters in 2006.

<table>
<thead>
<tr>
<th>Weight up to</th>
<th>60 g</th>
<th>100 g</th>
<th>150 g</th>
<th>200 g</th>
</tr>
</thead>
<tbody>
<tr>
<td>Postage</td>
<td>25 p</td>
<td>38 p</td>
<td>47 p</td>
<td>57 p</td>
</tr>
</tbody>
</table>

Postage, \( P = \begin{cases} 
25 & \text{for } 0 < W \leq 60 \\
38 & \text{for } 60 < W \leq 100 \\
47 & \text{for } 100 < W \leq 150 \\
57 & \text{for } 150 < W \leq 200 
\end{cases} \)
The range of the function weight-postage is the set of numbers \{25, 38, 47, 57\}. This is the graph:

![Graph of weight-postage function]

The “jumps” of this graph are called discontinuities of the function.

Graphs of discontinuous functions cannot be drawn without lifting the pencil from the paper.

**Exercise 8**

The price of a photocopy is 0.40 €. Draw the graph of the function number of photocopies - cost. Can you join the points?

**Exercise 9**

Draw the graph of the function \( y = \frac{1}{x} \). Is this a continuous function?

### 7.5.- ANALYTICAL EXPRESSION OF A FUNCTION

Lots of functions can be given by formulas. Remember that a formula describes the relationship between two or more variables.

**Example 1:**

Imagine you have a rope that is 80 cm long. If you join both ends, you can make countless rectangles.

**Example 2:**

The volume of a sphere is a function of its radius: \( V = \frac{4}{3} \pi r^3 \).

**Example 3:**

The area of a square is a function of its side: \( A = l^2 \).
Since the perimeter of these rectangles must be 80 cm, the base and the height will be $x$ and $40 - x$ respectively.

The table below shows the area of the rectangle for different values of the base:

<table>
<thead>
<tr>
<th>Base (cm)</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>35</th>
<th>$x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height (cm)</td>
<td>30</td>
<td>25</td>
<td>20</td>
<td>5</td>
<td>$40 - x$</td>
</tr>
<tr>
<td>Area (cm$^2$)</td>
<td>300</td>
<td>375</td>
<td>400</td>
<td>175</td>
<td>$x(40 - x)$</td>
</tr>
</tbody>
</table>

The area of these rectangles is given by the formula $A = x(40 - x)$, with $0 < x < 40$. The area, $A$, is a function of the base, $x$.

This formula is the **analytical expression** of the function, that is, the algebraic expression that relates both variables.

**Exercise 10**

The weight of one pound is 0.45 kg. Obtain the analytical expression of the function that converts pounds in kg. Draw its graph.

**Exercise 11**

The fixed cost for a company to operate a certain plant is $3,000 per day. It also costs $4 for each unit produced in the plant. Express the daily cost $C$ of operating the plant as a function of the number $n$ of units produced.
**Exercise 12**

An architect designs a window such that it has the shape of a rectangle with a semicircle on top, as shown. (This kind of window is called a Norman window).

The architect wants the base of the window to be 10 cm less than the height of the rectangular part.

Express the perimeter \( P \) of the window as a function of the radius \( r \) of the circular part.

**Exercise 13**

The side of a square is 14 cm. Four isosceles right triangles are cut out from the corners.

a) Express the area of the remainder octagon as a function of \( x \).

b) What is the domain of this function?

**Exercise 14**

Express the area of the coloured part as a function of \( x \).

**Exercise 15**

Match each graph with one of the following analytical expressions:

1) \( y = x + 1 \)\hspace{1cm}2) \( y = x^3 \)\hspace{1cm}3) \( y = x^2 \)\hspace{1cm}4) \( y = -x + 1 \)